Externally Driven Nonlinear Oscillator, Painlevé Test, First Integrals and Lie Symmetries

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For arbitrary constants c_1, c_2 and an arbitrary smooth functions f the driven anharmonic oscillator $\mathrm{d}^2u/\mathrm{d}t^2 + c_1\,\mathrm{d}u/\mathrm{d}t + c_2 u + u^3 = f(t)$ cannot be solved in closed form. We apply the Painlevé test to obtain the constraint on the constants c_1, c_2 and the function f for which the equation passes the test. We also give the Lie symmetry vector field and first integrals for this equation.

For arbitrary constants c_1, c_2 and an arbitrary smooth function f the anharmonic oscillator

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + c_1\frac{\mathrm{d}u}{\mathrm{d}t} + c_2 u + u^3 = f(t) \tag{1}$$

cannot be solved in closed form.

We apply the Painlevé test [1, 2, 3] to obtain the constraint on the constants c_1, c_2 and the function f for which (1) passes the test. The constraint on c_1 and c_2 gives an algebraic equation and the constraint of f is a linear differential equation. We solve these equations and give the Lie point symmetry and the first integral for this special case of (1).

Let us first discuss the Painlevé test for (1). A remark is in order for applying the Painlevé test for non-autonomous systems. The coefficients that depend on the independent variable must themselves be expanded in terms of $t-t_1$, where t_1 is the pole position and we use the identity $t \equiv (t-t_1)+t_1$. If nonautonomous terms enter the equation at lower order than the dominant balance the above mentioned expansion turns out to be unnecessary whereas if the nonautonomous terms are at dominant balance level they must be expanded with respect to $t-t_1$. We assume that f does not enter the expansion at dominant level.

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Before we study (1) we give a brief review of the special case

$$\frac{\mathrm{d}^2u}{\mathrm{d}t^2} + c_1\frac{\mathrm{d}u}{\mathrm{d}t} + c_2 u + u^3 = 0, \tag{2}$$

where c_1 and c_2 are constants. Equation (2) is considered in the complex domain with c_1 and c_2 real. For the sake of simplicity we do not change the notation. Inserting the Laurent expansion

$$u(t) = \sum_{j=0}^{\infty} a_j (t-t_1)^{j-n}, \tag{3}$$

where t_1 denotes the pole position, yields $n=1$ and $a_0^2 = -2$. The expansion coefficients a_1, a_2, and a_3 are determined by

$$3a_1a_0 = c_1, \quad 3a_2a_0 = -c_2 - 3a_1^2,$$
$$4a_3 = c_1a_2 + c_2a_1 + a_1^3 + 6a_0a_1a_2. \tag{4}$$

The expansion coefficient a_4 is arbitrary in expansion (3) if

$$c_1^2(2c_1^2 - 9c_2) = 0. \tag{5}$$

This means $r=4$ is a so-called resonance (compare [1, 2, 3] and references therein). The solution $c_1 = 0$ is the trivial case. To summarize: If $2c_1^2 = 9c_2$, then the general solution of (2) can be expressed in terms of Jacobi elliptic functions. For this case (i.e. $2c_1^2 = 9c_2$) we can find an explicitly time-dependent first integral, namely

$$I(t,u,\dot{u}) = \exp\left(\frac{4}{3}c_1 t\right)\left(\left(\dot{u} + \frac{c_1 u}{3}\right)^2 + \frac{1}{2}u^4\right). \tag{6}$$

If condition (5) is satisfied, then (2) admits two Lie symmetry vector fields

$$Z_1 = \frac{\partial}{\partial t}, \quad Z_2 = -\frac{c_1}{3}\exp\left(\frac{c_1 t}{3}\right)u\frac{\partial}{\partial u} + \exp\left(\frac{c_1 t}{3}\right)\frac{\partial}{\partial t}. \tag{7}$$

Let us now consider (1). Inserting the ansatz

$$u(t) = \sum_{j=0}^{\infty} u_j(t)\,\phi(t)^{j-n} \tag{8}$$

with $n=1$ into (1), we find at the resonance $r=4$ the condition

$$-27\sqrt{-2}\,\frac{\mathrm{d}f}{\mathrm{d}t} - 27\sqrt{-2}\,c_1 f - 18c_2c_1^2 + 4c_1^4 = 0. \tag{9}$$

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Since $\sqrt{-2}$ is imaginary and c_1 and c_2 are real, it follows that (9) decomposes into two equations, namely

$$\frac{df}{dt} + c_1 f = 0 \tag{10}$$

and condition (5). The general solution of (10) is given by

$$f(t) = C\, e^{-c_1 t}. \tag{11}$$

Consequently,

$$\frac{d^2u}{dt^2} + c_1 \frac{du}{dt} + \tfrac{2}{9} c_1^2 u + u^3 = C \exp(-c_1 t) \tag{12}$$

passes the Painlevé test. Equation (12) admits one Lie symmetry vector field, namely Z_2 given by (7). Now (12) can be derived from a Lagrangian function

$$\mathscr{L}(u, \dot{u}, t) = e^{c_1 t}\left(\tfrac{1}{2}\dot{u}^2 - V(u, t)\right), \tag{13}$$

where

$$V(u, t) = \tfrac{1}{9} c_1^2 u^2 + \tfrac{1}{4} u^4 - C u\, e^{-c_1 t}. \tag{14}$$

Thus we can apply Noether's theorem to find a first integral from the Lie symmetry vector field Z_2. We obtain

$$I(t, u, \dot{u}) = \exp\left(\frac{4}{3} c_1 t\right)\left(\left(\dot{u} + \frac{c_1 u}{3}\right)^2 + \frac{1}{2} u^4 - 2 C u\, e^{-c_1 t}\right). \tag{15}$$

We used REDUCE [4] and $C++$ [5] for most of the calculations performed in this paper.

[1] W.-H. Steeb and N. Euler, Nonlinear evolution equations and Painlevé test. World Scientific Publishing, Singapore 1988.
[2] N. Euler and W.-H. Steeb, Continuous symmetries, Lie algebras and differential equations, Bibliographisches Institut, Mannheim 1992.
[3] W.-H. Steeb, Invertible point transformation and nonlinear differential equations, World Scientific Publishing, Singapore 1993.
[4] W.-H. Steeb and D. Lewien, Algorithm and computations with REDUCE, Bibliographisches Institut, Mannheim 1992.
[5] W.-H. Steeb, D. Lewien, and O. Boine-Frankenheim: Object-oriented programming in science with C++, Bibliographisches Institut, Mannheim 1993.